Vector and Matrix Norms

We first present a brief overview of vector and matrix norms because they are useful in the discussion of errors and in the stopping criteria for iterative methods. Norms can be defined on any vector space, but we usually use $\mathbb{R}^n$ or $\mathbb{C}^n$. A vector norm $||\boldsymbol{x}||$ can be thought of as the **length** or **magnitude** of a vector $\boldsymbol{x} \in \mathbb{R}^n$. A **vector norm** is any mapping from $\mathbb{R}^n$ to $\mathbb{R}$ that obeys these three properties:

$$
\begin{aligned}
||x|| &> 0 \text{ if } \boldsymbol{x} \neq \boldsymbol{0} \\
||\alpha \boldsymbol{x}|| &= |\alpha|\,||\boldsymbol{x}|| \\
||\boldsymbol{x} + \boldsymbol{y}|| &\leqq ||\boldsymbol{x}|| + ||\boldsymbol{y}|| \qquad \textbf{(triangle inequality)}
\end{aligned}
$$

for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and scalars $\alpha \in \mathbb{R}$. Examples of vector norms for the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ are

$$
\begin{aligned}
||\boldsymbol{x}||_1 &= \sum_{i=1}^{n} |x_i| && \ell_1\textbf{-vector norm} \\
||\boldsymbol{x}||_2 &= \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} && \textbf{Euclidean/}\ell_2\textbf{-vector norm} \\
||\boldsymbol{x}||_\infty &= \max_{1 \leqq i \leqq n} |x_i| && \ell_\infty\textbf{-vector norm}
\end{aligned}
$$

For $n \times n$ matrices, we can also have **matrix norms**, subject to the same requirements:

$$
\begin{aligned}
&||\boldsymbol{A}|| > 0 \text{ if } \boldsymbol{A} \neq 0 \\
&||\alpha \boldsymbol{A}|| = |\alpha|\,||\boldsymbol{A}|| \\
&||\boldsymbol{A} + \boldsymbol{B}|| \leqq ||\boldsymbol{A}|| + ||\boldsymbol{B}|| \qquad \textbf{(triangular inequality)}
\end{aligned}
$$

for matrices $\boldsymbol{A}, \boldsymbol{B}$ and scalars α.

We usually prefer matrix norms that are related to a vector norm. For a vector norm $||\cdot||$, the **subordinate matrix norm** is defined by

$$
||\boldsymbol{A}|| \equiv \sup\{||\boldsymbol{A}\boldsymbol{x}|| : \boldsymbol{x} \in \mathbb{R}^n \text{ and } ||\boldsymbol{x}|| = 1\}
$$

Here, $\boldsymbol{A}$ is an $n \times n$ matrix. For a subordinate matrix norm, some additional properties are

$$
\begin{aligned}
||\boldsymbol{I}|| &= 1 \\
||\boldsymbol{A}\boldsymbol{x}|| &\leqq ||\boldsymbol{A}||\,||\boldsymbol{x}|| \\
||\boldsymbol{A}\boldsymbol{B}|| &\leqq ||\boldsymbol{A}||\,||\boldsymbol{B}||
\end{aligned}
$$

There are two meanings associated with the notation $||\cdot||_p$, one for vectors and another for matrices. The context will determine which one is intended. Examples of subordinate matrix norms for an $n \times n$ matrix $\boldsymbol{A}$ are

$$
\begin{aligned}
||\boldsymbol{A}||_1 &= \max_{1 \leqq j \leqq n} \sum_{i=1}^{n} |a_{ij}| && \ell_1\textbf{-matrix norm} \\
||\boldsymbol{A}||_2 &= \max_{1 \leqq i \leqq n} \sqrt{|\sigma_{\max}|} && \textbf{spectral /}\ell_2\textbf{-matrix norm} \\
||\boldsymbol{A}||_\infty &= \max_{1 \leqq i \leqq n} \sum_{j=1}^{n} |a_{ij}| && \ell_\infty\textbf{-matrix norm}
\end{aligned}
$$

Here, σ_i are the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$, which are called the **singular values** of $\boldsymbol{A}$. The largest $\sigma_{\max}$ in absolute value is termed the **spectral radius** of $\boldsymbol{A}$. (See Section 8.3 for a discussion of singular values.)

Condition Number and Ill-Conditioning

An important quantity that has some influence in the numerical solution of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is the **condition number**, which is defined as

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|_2 \, \|\boldsymbol{A}^{-1}\|_2$$

It turns out that it is not necessary to compute the inverse of $\boldsymbol{A}$ to obtain an estimate of the condition number. Also, it can be shown that the condition number $\kappa(\boldsymbol{A})$ gauges the transfer of error from the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ to the solution $\boldsymbol{x}$. The rule of thumb is that if $\kappa(\boldsymbol{A}) = 10^k$, then one can expect to lose at least k digits of precision in solving the system $\boldsymbol{Ax} = \boldsymbol{b}$. If the linear system is sensitive to perturbations in the elements of $\boldsymbol{A}$, or to perturbations of the components of $\boldsymbol{b}$, then this fact is reflected in $\boldsymbol{A}$ having a large condition number. In such a case, the matrix $\boldsymbol{A}$ is said to be **ill-conditioned**. Briefly, the larger the condition number, the more ill-conditioned the system.

Suppose we want to solve an invertible linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ for a given coefficient matrix $\boldsymbol{A}$ and right-hand side $\boldsymbol{b}$ but there may have been perturbations of the data owing to uncertainty in the measurements and roundoff errors in the calculations. Suppose that the right-hand side is perturbed by an amount assigned the symbol $\delta\boldsymbol{b}$ and the corresponding solution is perturbed an amount denoted by the symbol $\delta\boldsymbol{x}$. Then we have

$$\boldsymbol{A}(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{Ax} + \boldsymbol{A}\delta\boldsymbol{x} = \boldsymbol{b} + \delta\boldsymbol{b}$$

where

$$\boldsymbol{A}\delta\boldsymbol{x} = \delta\boldsymbol{b}$$

From the original linear system $\boldsymbol{Ax} = \boldsymbol{x}$ and norms, we have

$$||\boldsymbol{b}|| = ||\boldsymbol{Ax}|| \leqq ||\boldsymbol{A}|| \, ||\boldsymbol{x}||$$

which gives us

$$\frac{1}{||\boldsymbol{x}||} \leqq \frac{||\boldsymbol{A}||}{||\boldsymbol{b}||}$$

From the perturbed linear system $\boldsymbol{A}\delta\boldsymbol{x} = \delta\boldsymbol{b}$, we obtain $\delta\boldsymbol{x} = \boldsymbol{A}^{-1}\delta\boldsymbol{b}$ and

$$||\delta\boldsymbol{x}|| \leqq ||\boldsymbol{A}^{-1}|| \, ||\delta\boldsymbol{b}||$$

Combining the two inequalities above, we obtain

$$\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} \leqq \kappa(\boldsymbol{A}) \frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||}$$

which contains the condition number of the original matrix $\boldsymbol{A}$.

As an example of an ill-conditioned matrix consider the Hilbert matrix

$$H_3 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

We can use the Matlab commands to generate the matrix and then to compute both the condition number using the 2-norm and the determinant of the matrix. We find the condition number to be 524.0568 and the determinant to be 4.6296×10^{-4}. In solving linear systems, the condition number of the coefficient matrix measures the sensitivity of the system to errors in the data. When the condition number is large, the computed solution of the system may be dangerously in error! Further checks should be made before accepting the solution as being accurate. Values of the condition number near 1 indicate a well-conditioned matrix whereas large values indicate an ill-conditioned matrix. Using the determinant to check for singularity is appropriate only for matrices of modest size. Using mathematical software, one can compute the condition number to check for singular or near-singular matrices.

A goal in the study of numerical methods is to acquire an awareness of whether a numerical result can be trusted or whether it may be suspect (and therefore in need of further analysis). The condition number provides some evidence regarding this question. With the advent of sophisticated mathematical software systems such as Matlab and others, an estimate of the condition number is often available, along with an approximate solution so that one can judge the trustworthiness of the results. In fact, some solution procedures involve advanced features that depend on an estimated condition number and may switch solution techniques based on it. For example, this criterion may result in a switch of the solution technique from a variant of Gaussian elimination to a least-squares solution for an ill-conditioned system. Unsuspecting users may not realize that this has happened unless they look at all of the results, including the estimate of the condition number. (Condition numbers can also be associated with other numerical problems, such as locating roots of equations.)