Orthogonality - Sine and Cosine Integrals for Fourier Series

For any $n \neq 0$ and with $\mu_n = \dfrac{n\pi}{\ell}$ we have

1. $\displaystyle \int_{-\ell}^{\ell} \cos(\mu_n x)\, dx = \left[\frac{\sin(\mu_n x)}{\mu_n}\right]_{-\ell}^{\ell} = 0$

2. $\displaystyle \int_{-\ell}^{\ell} \sin(\mu_n x)\, dx = \left[-\frac{\cos(\mu_n x)}{\mu_n}\right]_{-\ell}^{\ell} = 0$

3. $\displaystyle \int_{-\ell}^{\ell} \cos^2(\mu_n x)\, dx = 2\int_{0}^{\ell} \left[\frac{1+\cos(2\mu_n x)}{2}\right] dx = \left[x + \frac{\sin(2\mu_n x)}{2\mu_n}\right]_{0}^{\ell} = \ell$

4. $\displaystyle \int_{-\ell}^{\ell} \sin^2(\mu_n x)\, dx = 2\int_{0}^{\ell} \left[\frac{1-\cos(2\mu_n x)}{2}\right] dx = \left[x - \frac{\sin(2\mu_n x)}{2\mu_n}\right]_{0}^{\ell} = \ell$

The main three formulas follow from the trig identities

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right),$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right),$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right).$$

For $n \neq m$ we have

1. $\displaystyle \int_{-\ell}^{\ell} \cos(\mu_n x)\cos(\mu_m x)\, dx = \frac{1}{2}\int_{-\ell}^{\ell} [\cos((\mu_n+\mu_m)x) + \cos((\mu_n-\mu_m)x])\ dx = 0$

2. $\displaystyle \int_{-\ell}^{\ell} \sin(\mu_n x)\sin(\mu_m x)\, dx = \frac{1}{2}\int_{-\ell}^{\ell} [\cos((\mu_n-\mu_m)x) - \cos((\mu_n-\mu_m)x)]\ dx = 0$

3. $\displaystyle \int_{-\ell}^{\ell} \sin(\mu_n x)\cos(\mu_m x)\, dx = \frac{1}{2}\int_{-\ell}^{\ell} [\sin((\mu_n+\mu_n)x) + \sin((\mu_n-\mu_n)x)]\ dx = 0$