Trigonometric Interpolation

Given a function $f(x)$ defined in the interval $0 \leqq x \leqq 2\pi$ and given an integer N we seek a trigonometric polynomial $S(x)$ such that

13.20 $$S(x_k) = \hat{f}(x_k), \qquad k = 0, 1, \ldots, N$$

where x_k is given by (13.15). Here, for convenience, we define the function $\hat{f}(x)$ by

13.21 $$\hat{f}(x) = \begin{cases} f(x), & x \neq 0, 2\pi \\ \dfrac{f(0) + f(2\pi)}{2} & x = 0, 2\pi. \end{cases}$$

In the case N is even, we let $S(x)$ have the form $S_M^*(x)$ given by (13.18) where

13.22 $$M = \frac{N}{2}.$$

On the other hand, if N is odd, then we let $S(x) = S_M(x)$ where

13.23 $$S_M(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{M} (a_n \cos nx + b_n \sin nx)$$

where

13.24 $$M = \frac{N-1}{2}.$$

We first seek to show the existence of a unique solution of (13.20). Let us consider the case where N is even. Evidently by (13.18) we have N equations and N unknowns $a_0, a_1, \ldots, a_M$ and $b_1, b_2, \ldots, b_{M-1}$. We can show that this system has a unique solution if we can show that its determinant does not vanish. To show this, it is, by Theorem A-7.8, sufficient to show that the homogeneous system obtained from (13.20) by letting $\hat{f}(x) \equiv 0$ has only the trivial solution

13.25 $$a_0 = a_1 = \cdots = a_M = b_1 = b_2 = \cdots = b_{M-1} = 0.$$

We now prove

13.26 Lemma.

13.27 $$\sum_{k=1}^{N} \cos nx_k = \begin{cases} 0, & \text{if } n/N \text{ is not an integer,} \\ N, & \text{if } n/N \text{ is an integer,} \end{cases}$$

and

13.28 $$\sum_{k=1}^{N} \sin nx_k = 0.$$

Here the x_k are given by (13.15).

Proof. Evidently we have

13.29
$$\sum_{k=1}^{N} \cos nx_k + i \sum_{k=1}^{N} \sin nx_k = \sum_{k=1}^{N} e^{inx_k} = \sum_{k=1}^{N} e^{inkh}.$$

This is a geometric progression with ratio e^{inh}. Since $e^{inh} = 1$ if and only if $nh = 2\pi p$ for some integer p, i.e., if and only if n/N is an integer, it follows that

13.30
$$\sum_{k=1}^{N} e^{inkh} = \begin{cases} N, \text{if } n/N \text{ is an integer} \\ \dfrac{e^{inh} - e^{i(N+1)nh}}{1 - e^{inh}}, & \text{otherwise.} \end{cases}$$

But since $e^{iNnh} = 1$ the lemma follows.

We now prove

13.31 Lemma. If x_k is given for (13.15), then

13.32
$$\sum_{k=1}^{N} \cos nx_k \cos mx_k = \begin{cases} 0, & \text{if neither } \dfrac{n+m}{N} \text{ nor } \dfrac{n-m}{N} \text{ is an integer} \\ \dfrac{N}{2}, & \text{if either } \dfrac{n+m}{N} \text{ or } \dfrac{n-m}{N} \text{ but not both is an integer} \\ N, & \text{if both } \dfrac{n+m}{N} \text{ and } \dfrac{n-m}{N} \text{ are integers} \end{cases}$$

13.33
$$\sum_{k=1}^{N} \cos nx_k \sin mx_k = 0$$

13.34
$$\sum_{k=1}^{N} \sin nx_k \sin mx_k = \begin{cases} 0, & \text{if neither } \dfrac{n+m}{N} \text{ nor } \dfrac{n-m}{N} \text{ is an integer or if both } \dfrac{n+m}{N} \text{ and } \dfrac{n-m}{N} \text{ are integers} \\ -\dfrac{N}{2}, & \text{if } \dfrac{m+n}{N} \text{ is an integer and } \dfrac{m-n}{N} \text{ is not} \\ \dfrac{N}{2}, & \text{if } \dfrac{m-n}{N} \text{ is an integer and } \dfrac{m+n}{N} \text{ is not.} \end{cases}$$

Proof. The lemma follows from Lemma 13.26 and from the identities

13.35
$$\begin{cases} \cos nx_k \cos mx_k = \frac{1}{2}\cos(n+m)x_k + \frac{1}{2}\cos(n-m)x_k \\ \cos nx_k \sin mx_k = \frac{1}{2}\sin(m+n)x_k - \frac{1}{2}\sin(m-n)x_k \\ \sin nx_k \sin mx_k = \frac{1}{2}\cos(n-m)x_k - \frac{1}{2}\cos(n+m)x_k \end{cases}$$

Suppose now that (13.20) holds where $S(x)$ has the form $S_M^*(x)$ given by (13.18). Then by Lemma 13.31 we have for $n = 0, 1, \ldots, M$

13.36
$$\sum_{k=1}^{N} \hat{f}(x_k) \cos nx_k = \sum_{k=1}^{N} \left(\tfrac{1}{2}a_0 + \sum_{m=1}^{M-1} (a_m \cos mx_k + b_m \sin mx_k) + \tfrac{1}{2}a_M \cos Mx_k \right) \cos nx_k = \frac{N}{2} a_n.$$

Moreover, we also have for $n = 1, 2, \ldots, M - 1$

13.37
$$\sum_{k=1}^{N} \hat{f}(x_k) \sin nx_k = \frac{N}{2} b_n.$$

Thus, we have

13.38
$$\begin{cases} a_n = \dfrac{2}{N} \displaystyle\sum_{k=1}^{N} \hat{f}(x_k) \cos nx_k, & n = 0, 1, \ldots, M \\ b_n = \dfrac{2}{N} \displaystyle\sum_{k=1}^{N} \hat{f}(x_k) \sin nx_k, & n = 1, 2, \ldots, M - 1. \end{cases}$$

If $\hat{f}(x) \equiv 0$ then all the a_n and b_n vanish. Hence the determinant of the system (13.20) does not vanish and (13.20) has a unique solution. Since a solution, if it exists, must be given by (13.38) it follows that (13.38) is indeed the solution.

In the case N is odd, a similar analysis holds and we get

13.39
$$\begin{cases} a_n = \dfrac{2}{N} \displaystyle\sum_{k=1}^{N} \hat{f}(x_k) \cos nx_k, & n = 0, 1, 2, \ldots, M \\ b_n = \dfrac{2}{N} \displaystyle\sum_{k=1}^{N} \hat{f}(x_k) \sin nx_k, & n = 1, 2, \ldots, M. \end{cases}$$

13.40 Theorem. Let $f(x)$ be defined on the interval $0 \leqq x \leqq 2\pi$ and let $\hat{f}(x)$ be given by (13.21). If N is even and if $S(x) = S_M^*(x)$, where $S_M^*(x)$ is given by (13.18) and where the coefficients a_n and b_n are given by (13.38), then

13.41
$$S(x_k) = \hat{f}(x_k), \qquad k = 0, 1, \ldots, N$$

where x_k is given by (13.15). If N is odd and if $S(x) = S_M(x)$, where $S_M(x)$ is given by (13.23) and where the coefficients a_n and b_n are given by (13.39), then (13.41) holds. Conversely, if (13.41) holds for some trigonometric polynomial of the form (13.18), if N is even, and (13.23) if N is odd, then the coefficients a_n and b_n are given by (13.38) if N is even, and by (13.39) if N is odd.

As an example, consider the case $f(x) = x$, $N = 4$. Evidently the Fourier series for $f(x)$ is given by

13.42 $$f(x) = \pi - 2\sin x - \sin 2x - \tfrac{2}{3}\sin 3x - \tfrac{1}{2}\sin 4x \ldots .$$

Moreover, since $1 - (\frac{1}{3}) + (\frac{1}{5}) - (\frac{1}{7}) + (\frac{1}{9}) + \cdots = \pi/4$ we have

13.43 $$\begin{cases} \hat{f}(x_0) = \hat{f}(x_4) = \frac{1}{2}(2\pi) = \pi \\ \hat{f}(x_1) = f(x_1) = \dfrac{\pi}{2} \\ \hat{f}(x_2) = f(x_2) = \pi \\ \hat{f}(x_3) = f(x_3) = \dfrac{3\pi}{2} . \end{cases}$$

By (13.22) $M = 2$ and

13.44 $$S_2^*(x) = \tfrac{1}{2}a_0 + a_1\cos x + \tfrac{1}{2}a_2\cos 2x + b_1 \sin x.$$

By (13.38) we have

13.45 $$\begin{cases} a_0 = \dfrac{2}{N}\sum_{k=1}^{4} \hat{f}(x_k) = \dfrac{2}{N}(4\pi) = 2\pi \\ a_1 = \dfrac{2}{N}\left\{\dfrac{\pi}{2}\cos\dfrac{2\pi}{N} + \pi\cos\dfrac{4\pi}{N} + \dfrac{3\pi}{2}\cos\dfrac{6\pi}{N} + \pi\cos\dfrac{8\pi}{N}\right\} = 0 \\ a_2 = \dfrac{2}{N}\left\{\dfrac{\pi}{2}\cos\dfrac{4\pi}{N} + \pi\cos\dfrac{8\pi}{N} + \dfrac{3\pi}{2}\cos\dfrac{12\pi}{N} + \pi\cos\dfrac{16\pi}{N}\right\} = 0 \\ b_1 = \dfrac{2}{N}\left\{\dfrac{\pi}{2}\sin\dfrac{2\pi}{N} + \pi\sin\dfrac{4\pi}{N} + \dfrac{3\pi}{2}\sin\dfrac{6\pi}{N} + \pi\sin\dfrac{8\pi}{N}\right\} = -\dfrac{\pi}{2}. \end{cases}$$

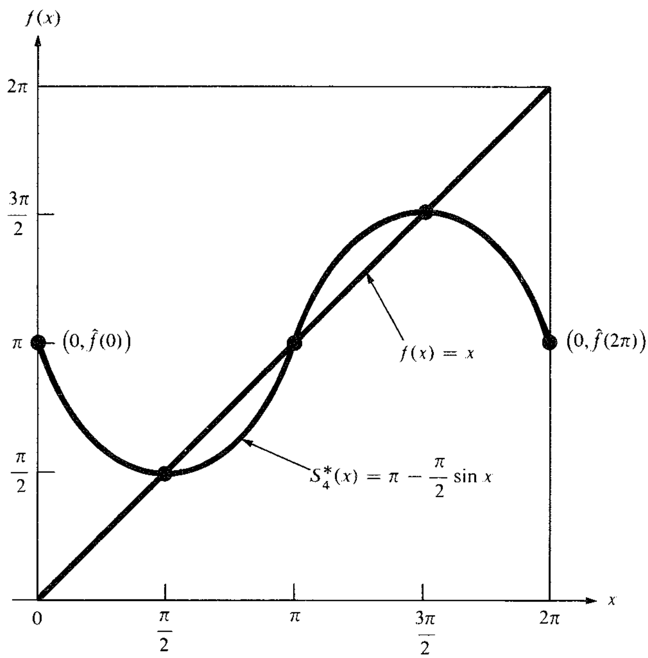
Therefore the interpolating trigonometric polynomial is

13.46 $$S_4^*(x) = \pi - \frac{\pi}{2}\sin x.$$

We verify that

13.47
$$\begin{cases} S_4^*(0) &= \pi = \hat{f}(0) \\ S_4^*\left(\dfrac{\pi}{2}\right) &= \dfrac{\pi}{2} = \hat{f}\left(\dfrac{\pi}{2}\right) \\ S_4^*(\pi) &= \pi = \hat{f}(\pi) \\ S_4^*\left(\dfrac{3\pi}{2}\right) &= \dfrac{3\pi}{2} = \hat{f}\left(\dfrac{3\pi}{2}\right) \\ S_4^*(2\pi) &= \pi = \hat{f}(2\pi)\,. \end{cases}$$

Hence the required conditions are satisfied.



13.48 Fig. Trigonometric Interpolation for $f(x) = x$, $0 \leqq x \leqq 2\pi$.

It can be seen from Fig. 13.48 that the agreement between $S_4^*(x)$ and $f(x)$ is fairly good except near the ends of the interval $0 \leqq x \leqq 2\pi$. If the function were continuous and periodic with period 2π, this difficulty would not arise since we would have $f(0) = f(2\pi)$ and $\hat{f}(0) = f(0) = \hat{f}(2\pi) = f(2\pi)$.

We remark that if for all x in the interval $0 \leqq x \leqq 2\pi$ we have, for N even and $M = N/2$, 13.49 $$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

then

13.50 $$a_n = \alpha_n + \alpha_{N+n} + \alpha_{2N+n} + \cdots + \alpha_{N-n} + \alpha_{2N-n} + \cdots, \quad n = 0, 1, 2, \ldots, M$$

and

13.51 $$b_n = \beta_n + \beta_{N+n} + \beta_{2N+n} + \cdots - \beta_{N-n} - \beta_{2N-n} - \cdots, \quad n = 1, 2, \ldots, M - 1.$$

Thus it can be seen that if N is large and if the coefficients α_k and β_k tend to zero reasonably rapidly, then the coefficients a_n and b_n are good approximations to the α_n and the β_n, respectively. (We remark that although it is possible for (13.49) to hold even if the α_n and β_n are not the Fourier coefficients for $f(x)$, nevertheless, if the convergence is uniform, then the α_n and β_n must be the Fourier coefficients.)

In our example the Fourier coefficients are given by

13.52 $$\alpha_0 = 2\pi, \qquad \alpha_1 = \alpha_2 = \cdots = 0, \qquad \beta_k = \frac{-2}{k} \qquad k = 1, 2, \ldots$$

and

13.53 $$a_0 = \alpha_0 + \alpha_4 + \alpha_8 + \cdots$$
$$+ \alpha_4 + \alpha_8 + \cdots = 2\pi$$

13.54 $$a_1 = a_2 = \cdots = 0$$

13.55 $$\begin{aligned} b_1 &= \beta_1 + \beta_5 + \beta_9 + \cdots \\ &\quad - \beta_3 - \beta_7 - \beta_{11} - \cdots \\ &= (-2)(1 + \tfrac{1}{5} + \tfrac{1}{7} + \cdots - \tfrac{1}{3} - \tfrac{1}{7} - \tfrac{1}{11} - \cdots) \\ &= (-2)\left(\frac{\pi}{4}\right) = -\frac{\pi}{2} \end{aligned}$$

which is consistent with (13.52) and with (13.50)—(13.51). Thus (13.50) and (13.51) hold even though the series does not converge to $f(x)$ at $x = 0$ and 2π. (However, the series does converge to $\hat{f}(x)$ at $x = 0$ and 2π, though the convergence is not uniform and $\hat{f}(x)$ is not continuous.)

Least Squares Trigonometric Approximation Over a Finite Point Set

We now show that if we truncate the interpolating trigonometric polynomial (13.18) or (13.23) we get a polynomial

13.56 $$S_L(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{L} (a_n \cos nx + b_n \sin nx)$$

which has the property that

13.57 $$\sum_{k=1}^{N} (\hat{f}(x_k) - S_L(x_k))^2 \leqq \sum_{k=1}^{N} (\hat{f}(x_k) - \hat{S}_L(x_k))^2$$

where $\hat{S}_L(x_k)$ is any other trigonometric polynomial of degree L or less of the form

13.58 $$\hat{S}_L(x_k) = \frac{\hat{a}_0}{2} + \sum_{n=1}^{L} (\hat{a}_n \cos nx + \hat{b}_n \sin nx).$$

We assume that $L < N/2$, and we let $\hat{f}(x)$ be given by (13.21). Evidently, we have

13.59 $$\sum_{k=1}^{N} (\hat{f}(x_k) - \hat{S}_L(x_k))^2 = \sum_{k=1}^{N} \hat{f}(x_k)^2 - \frac{N}{2} a_0\hat{a}_0 - \sum_{n=1}^{L} N(a_n\hat{a}_n + b_n\hat{b}_n) + \frac{N}{4}\hat{a}_0^2 + \sum_{n=1}^{L} \frac{N}{2}(\hat{a}_n^2 + \hat{b}_n^2)$$
$$= \sum_{k=1}^{N} \hat{f}(x_k)^2 + \frac{N}{4}(\hat{a}_0 - a_0)^2 + \frac{N}{2}\sum_{n=1}^{L} [(\hat{a}_n - a_n)^2 + (\hat{b}_n - b_n)^2] - \frac{N}{4} a_0^2 - \frac{N}{2}\left(\sum_{n=1}^{L} (a_n^2 + b_n^2)\right)$$

which is clearly minimized when

13.60 $$\begin{cases} \hat{a}_n = a_n, & n = 0, 1, 2, \ldots, L \\ \hat{b}_n = b_n, & n = 1, 2, \ldots, L \end{cases}$$

Therefore $S_L(x)$ is the trigonometric polynomial of best approximation in the sense of (13.57).