## §1. Notations and definitions.

Consider a function $f$ defined on a subset $G$ of the complex plane. Let $\{x_i\}_{i\in\mathbf{N}}$ be a sequence of different points belonging to $G$. We still denote the exact degree of a polynomial $p$ by $\partial p$ and its order by $\omega p$. The **rational interpolation problem** of order $(m, n)$ for $f$ consists in finding polynomials

$$p(x) = \sum_{i=0}^{m} a_i x^i$$

and

$$q(x) = \sum_{i=0}^{n} b_i x^i$$

with $p(x)/q(x)$ irreducible and such that

$$f(x_i) = \frac{p}{q}(x_i) \qquad i = 0, \ldots, m+n \qquad \textbf{(3.1.)}$$

Instead of solving problem (3.1.) we consider the linear system of equations

$$f(x_i)q(x_i) - p(x_i) = 0 \qquad i = 0, \ldots, m+n \qquad \textbf{(3.2.)}$$

Condition (3.2.) is a homogeneous system of $m+n+1$ linear equations in the $m+n+2$ unknown coefficients $a_i$ and $b_i$ of $p$ and $q$. Hence the system (3.2.) always has at least one nontrivial solution. For different solutions of (3.2.) the following equivalence can be proved.

**Theorem 3.1.**

If the polynomials $p_1$, $q_1$ and $p_2$, $q_2$ both satisfy (3.2.) then $p_1q_2 = p_2q_1$.

**Proof**

For the polynomial $p_1q_2 - p_2q_1$ we can write

$$(p_1q_2 - p_2q_1)(x_i) = [(fq_2 - p_2)q_1 - (fq_1 - p_1)q_2](x_i) = 0 \qquad i = 0, \ldots, m+n$$

Since $\partial(p_1 q_2 - p_2 q_1) \leq m+n$ it must vanish identically for it has more than $m+n$ zeros. ∎

Not all solutions of (3.2.) also satisfy (3.1.): it is very well possible that the polynomials $p$ and $q$ satisfying (3.2.) are such that $p/q$ is reducible. Nevertheless, because of theorem 3.1., all solutions of (3.2.) have the same irreducible form. For $p$ and $q$ satisfying (3.2.) we shall denote by

$$r_{m,n}(x) = \frac{p_0}{q_0}(x)$$

the irreducible form of $p/q$, where $q_0(x)$ is normalized such that $q_0(x_0) = 1$, and we shall call $r_{m,n}(x)$ the **rational interpolant** of order $(m,n)$ for $f$.
The following result is a consequence of theorem 3.1.

**Theorem 3.2.**

For every nonnegative $m$ and $n$ a unique rational interpolant of order $(m,n)$ for $f$ exists.

Although the terminology "interpolant" is used it may be that $r_{m,n}(x)$ does not satisfy the interpolation conditions (3.1.) anymore [14]. A simple example will illustrate this. Let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $f(x_0) = 0$, $f(x_1) = 3$, $f(x_2) = 3$. Take $m = n = 1$. Then the system of interpolation conditions is

$$\begin{cases} a_0 = 0 \\ 3(b_0 + b_1) - (a_0 + a_1) = 0 \\ 3(b_0 + 2b_1) - (a_0 + 2a_1) = 0 \end{cases}$$

A solution is $p(x) = 3x$ and $q(x) = x$. Thus $p_0(x) = 3$ and $q_0(x) = 1$. Clearly

$$\frac{p_0}{q_0}(x_0) \neq f(x_0)$$

Note the similarity with the Padé approximation problem: the Padé approximant of order $(m,n)$ did not necessarily satisfy condition (2.2.) anymore. We shall see that many properties valid for Padé approximants can be generalized for rational interpolants [20].

## §2. Fundamental properties.

### *2.1. Properties of the rational interpolant.*

Let $r_{m,n} = p_0/q_0$ be the rational interpolant of order $(m, n)$ for $f$. If $p_0$ and $q_0$ do not satisfy the system of conditions (3.2) themselves, it is easy to construct polynomials $p$ and $q$ from $p_0$ and $q_0$ that are a solution of (3.2.). Denote the exact degree of $p_0$ by $\mathbf{m}'$ and the exact degree of $q_0$ by $\mathbf{n}'$.

**Theorem 3.3.**

If the rational interpolant of order $(m, n)$ for $f$ is

$$r_{m,n}(x) = \frac{p_0}{q_0}(x)$$

then an integer $s$ exists with $0 \leq s \leq \min(m - m', n - n')$ and $s$ points $y_1, \ldots, y_s$ exist belonging to $\{x_0, \ldots, x_{m+n}\}$ such that

$$p(x) = p_0(x) \prod_{i=1}^{s} (x - y_i)$$

and

$$q(x) = q_0(x) \prod_{i=1}^{s} (x - y_i)$$

satisfy (3.2.).

**Proof**

Let $p_1(x)$ and $q_1(x)$ be a solution of (3.2.). Then

$$(f\, q_1)(x_i) = p_1(x_i) \quad i = 0, \ldots, m + n$$

Hence if $q_1(x_i) = 0$ also $p_1(x_i) = 0$. Let $\{y_1, \ldots, y_s\}$ be the set of zeros of $q_1(x)$ belonging to $\{x_0, \ldots, x_{m+n}\}$. We construct

$$t(x) = \prod_{i=1}^{s} (x - y_i)$$

If

$$r_{m,n}(x) = \frac{p_0}{q_0}(x)$$

then a polynomial $v$ exists with

$$p_1(x) = v(x)\,t(x)\,p_0(x)$$

$$q_1(x) = v(x)\,t(x)\,q_0(x)$$

Consequently

$$s = \partial t \leq min(m - m', n - n')$$

Since

$$q_1(x_i) \neq 0 \text{ for } x_i \in \{x_0, \ldots, x_{m+n}\} \backslash \{y_1, \ldots, y_s\}$$

we have

$$(fq_0)(x_i) = p_0(x_i) \quad \text{for} \quad x_i \in \{x_0, \ldots, x_{m+n}\} \backslash \{y_1, \ldots, y_s\}$$

If we put $p(x) = p_0(x)\,t(x)$ and $q(x) = q_0(x)\,t(x)$ then $p$ and $q$ also satisfy (3.2.). ∎

As a conclusion we can say that the rational interpolation problem (3.1.) has a solution if and only if $p_0(x)$ and $q_0(x)$ satisfy the system of equations (3.2.).

### *2.2. The table of rational interpolants.*

The rational interpolants of order $(m, n)$ for $f$ can again be ordered in a table:

$$\begin{array}{cccc} r_{0,0} & r_{0,1} & r_{0,2} & \cdots \\ r_{1,0} & r_{1,1} & r_{1,2} & \cdots \\ r_{2,0} & r_{2,1} & \cdots & \\ r_{3,0} & r_{3,1} & \cdots & \\ \vdots & \vdots & & \end{array}$$

In the first column one finds the polynomial interpolants for $f$ and in the first row the inverses of the polynomial interpolants for $\frac{1}{f}$. By theorem 3.3. at least