18.5 Toeplitz linear system solver.

The linear system of equations (18.20) can be solved in only order n^2 operations using the Levinson algorithm. When trying to exploit structure in a linear system of equations, the technique of pivoting as in Gaussian elimination should be avoided, because it destroys the structure, and replaced by another approach. The Levinson algorithm for Toeplitz systems constructs the solution recursively: the $k \times k$ Toeplitz system

$$\begin{cases} c_n b_1^{(k)} + \ldots + c_{n+1-k} b_k^{(k)} = -c_{n+1} \\ \quad \vdots \\ c_{n+k-1} b_1^{(k)} + \ldots + c_n b_k^{(k)} = -c_{n+k} \end{cases}$$

is solved for $k = 1, \ldots, m$ and the desired solution $(b_1, \ldots, b_m)$ of (18.20) is found in the final stage $k = m$.

Let us describe the procedure for a general nonsymmetric Toeplitz matrix as above and a general right hand side $(y_1, \ldots, y_m)$ instead of the particular right hand side $(-c_{n+1}, \ldots, -c_{n+m})$. When increasing k to $k+1$, the new linear system which contains an additional equation and one more unknown, is given by

$$\sum_{j=1}^{k} c_{n+i-j} b_j^{(k+1)} + c_{n+i-(k+1)} b_{k+1}^{(k+1)} = y_i \qquad i = 1, \ldots, k+1 \tag{18.22}$$

When subtracting the first k linear equations satisfied by the $b_j^{(k+1)}$ from these satisfied by the $b_j^{(k)}$, we find

$$\sum_{j=1}^{k} c_{n+i-j} \frac{b_j^{(k)} - b_j^{(k+1)}}{b_{k+1}^{(k+1)}} = c_{n+i-(k+1)} \qquad i = 1, \ldots, k$$

After introducing the notation

$$B_{k+1-j}^{(k)} = \frac{b_j^{(k)} - b_j^{(k+1)}}{b_{k+1}^{(k+1)}}$$

the linear system looks like

$$\sum_{j=1}^{k} c_{n+i-j} B_{k+1-j}^{(k)} = c_{n+i-(k+1)} \qquad i = 1, \ldots, k \tag{18.23}$$

where

$$b_j^{(k+1)} = b_j^{(k)} + B_{k+1-j}^{(k)} b_{k+1}^{(k+1)} \qquad j = 1, \ldots, k \tag{18.24}$$

So we have to focus on the computation of $b_{k+1}^{(k+1)}$ and the values $B_{k+1-j}^{(k)}$. For $b_{k+1}^{(k+1)}$ we can use the remaining $(k+1)^{th}$ equation of (18.22). When substituting here-in $b_j^{(k+1)}$ by expression (18.24) we find

$$\sum_{j=1}^{k} c_{n+k+1-j} b_j^{(k)} - y_{k+1} = b_{k+1}^{(k+1)} \left(\sum_{j=1}^{k} B_{k+1-j}^{(k)} c_{n+k+1-j} - c_n \right)$$

or

$$b_{k+1}^{(k+1)} = \frac{\sum_{j=1}^{k} c_{n+k+1-j} b_j^{(k)} - y_{k+1}}{\sum_{j=1}^{k} c_{n+k+1-j} B_{k+1-j}^{(k)} - c_n} \tag{18.25}$$

In this expression for $b_{k+1}^{(k+1)}$ the $B_{k+1-j}^{(k)}$ are still unknown. Let us now concentrate on their computation. Consider the mirrored linear system

$$\sum_{j=1}^{m} c_{n+j-i} d_j = y_i \qquad i = 1, \ldots, m$$

where the coefficients c_{n-j} are replaced by c_{n+j}. In the same way as above we can rewrite this related but different linear system of equations as

$$D_{k+1-j}^{(k)} = \frac{d_j^{(k)} - d_j^{(k+1)}}{d_{k+1}^{(k+1)}}$$

$$\sum_{j=1}^{k} c_{n+j-i} D_{k+1-j}^{(k)} = c_{n+(k+1)-i} \qquad i = 1, \ldots, k \tag{18.26}$$

We renumber the equations and unknowns of (18.23) and (18.26) such that i and j run from k to 1 by introducing new indices i and j respectively replacing $k+1-i$ and $k+1-j$. In this way we obtain

$$\sum_{j=k}^{1} c_{n+j-i} B_j^{(k)} = c_{n-i} \qquad i = 1, \ldots, k$$

$$\sum_{j=k}^{1} c_{n+i-j} D_j^{(k)} = c_{n+i} \qquad i = 1, \ldots, k$$

Note that the values $D_j^{(k)}$ actually satisfy the same equations as the $b_j^{(k)}$ when replacing the right hand sides y_i by c_{n+i}. Hence, in the same way as (18.25) was obtained we can write

$$D_{k+1}^{(k+1)} = \frac{\sum_{j=1}^{k} c_{n+k+1-j} D_j^{(k)} - c_{n+k+1}}{\sum_{j=1}^{k} c_{n+k+1-j} B_{k+1-j}^{(k)} - c_n} \tag{18.27}$$

Analogously, the $B_j^{(k)}$ apparently satisfy the same system as the $d_j^{(k)}$ after replacing the right hand sides y_i by c_{n-i}. Taking also into account that in the mirrored system of equations c_{n+j} replaces c_{n-j} and $B_{k+1-j}^{(k)}$ becomes $D_{k+1-j}^{(k)}$, we again find in exactly the same way as for (18.25) that

$$B_{k+1}^{(k+1)} = \frac{\sum_{j=1}^{k} c_{n+j-(k+1)} B_j^{(k)} - c_{n-(k+1)}}{\sum_{j=1}^{k} c_{n+j-(k+1)} D_{k+1-j}^{(k)} - c_n} \tag{18.28}$$

Remember that

$$\begin{aligned} D_j^{(k+1)} &= D_j^{(k)} - B_{k+1-j}^{(k)} D_{k+1}^{(k+1)} \\ B_j^{(k+1)} &= B_j^{(k)} - D_{k+1-j}^{(k)} B D_{k+1}^{(k+1)} \end{aligned} \tag{18.29}$$

and

$$\begin{aligned} b_j^{(k+1)} &= b_j^{(k)} - B_{k+1-j}^{(k)} b_{k+1}^{(k+1)} \\ d_j^{(k+1)} &= d_j^{(k)} - D_{k+1-j}^{(k)} d_{k+1}^{(k+1)} \end{aligned} \tag{18.30}$$

In the end, when using the formulas (18.27), (18.28), (18.29) and (18.30) in that order with the starting values

$$\begin{aligned} b_1^{(1)} &= y_1/c_n \\ B_1^{(1)} &= c_{n-1}/c_n \\ D_1^{(1)} &= c_{n+1}/c_n \end{aligned}$$

the Levinson algorithm can be carried out for $k = 1, \ldots, m$.