3

Approximation operators and some approximating functions

3.1 Approximation operators

We continue to let $\mathscr{A}$ be a set of approximating functions in a normed linear space $\mathscr{B}$. It was noted in Section 2.3 that if, for every f in $\mathscr{B}$, there is a unique best approximation from $\mathscr{A}$ to f, $X(f)$ say, then we may regard X as an operator from $\mathscr{B}$ to $\mathscr{A}$. We now take the more general point of view that X is an approximation operator if it is any mapping from $\mathscr{B}$ to $\mathscr{A}$.

Nearly all numerical methods for calculating approximations are approximation operators. It is only necessary for the method to select a unique element of $\mathscr{A}$ as an approximation to any f in $\mathscr{B}$. We make this remark because it is helpful sometimes to relate some fundamental properties of operators to algorithms.

For example, some of the work of Chapter 17 concerns algorithms that possess the projection property. Therefore we note that the operator X is defined to be a projection if the equation

$$X[X(f)] = X(f), \qquad f \in \mathscr{B}, \tag{3.1}$$

is satisfied. Hence a sufficient condition for X to be a projection is the equation

$$X(a) = a, \qquad a \in \mathscr{A}. \tag{3.2}$$

Most of the approximation methods that are considered in this book do satisfy condition (3.2), but an important exception is the Bernstein operator, which is discussed in Chapter 6. Sometimes $X(f)$ is written as Xf.

The idea of a linear operator is also well known; namely, we define X to be linear if the equation

$$X(\lambda f) = \lambda X(f) \tag{3.3}$$

holds for all $f \in \mathscr{B}$, where λ is any real number, and if the equation

$$X(f+g) = X(f) + X(g) \tag{3.4}$$

is obtained for all $f \in \mathscr{B}$ and for all $g \in \mathscr{B}$. Usually, when X is linear and when $\mathscr{A}$ is a finite-dimensional linear space, the calculation of $X(f)$ reduces to the solution of a system of linear equations. For example, we find in Chapter 11 that this case occurs when $X(f)$ is the best approximation to f with respect to the 2-norm. However, if $X(f)$ is the best approximation in the 1-norm or ∞-norm, then X is hardly ever a linear operator.

Also we make frequent use of the norm of an approximation operator. The norm of X is written as $\|X\|$, and it is the smallest real number such that the inequality

$$\|X(f)\| \leq \|X\| \, \|f\| \tag{3.5}$$

holds for all $f \in \mathscr{B}$. The notation $\|X\|_p$ indicates that $\|X\|$ is derived from $\|f\|_p$.

An example of an approximation operator that is useful because it is easy to apply is as follows. Let $\mathscr{B}$ be the space $\mathscr{C}[0, 1]$ of real-valued functions that are continuous on $[0, 1]$, and let $\mathscr{A}$ be the linear space $\mathscr{P}_1$ of all real polynomials of degree at most one. Then, in order that the calculation of an approximation to a function f in $\mathscr{B}$ depends on only two function evaluations, we let p be the polynomial in $\mathscr{A}$ that satisfies the interpolation conditions

$$\left.\begin{aligned} p(0) &= f(0) \\ p(1) &= f(1) \end{aligned}\right\}. \tag{3.6}$$

Thus $p = X(f)$, where X is a linear projection operator from $\mathscr{B}$ to $\mathscr{A}$.

In order to define the norm of this operator we choose a norm for the space $\mathscr{C}[0, 1]$. However, if the 2-norm

$$\|f\|_2 = \left\{ \int_0^1 [f(x)]^2 \, dx \right\}^{\frac{1}{2}}, \qquad f \in \mathscr{C}[0, 1], \tag{3.7}$$

is used, we find that the operator X is unbounded, because it is possible for $\|Xf\|_2$ to be one when $\|f\|_2$ is arbitrarily small. It is therefore necessary to prefer the ∞-norm

$$\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|, \qquad f \in \mathscr{C}[0, 1], \tag{3.8}$$

when considering approximation operators that are defined by interpolation conditions. In this case, because p is in $\mathscr{P}_1$, equation (3.6) implies the inequality

$$\begin{aligned}\|X(f)\| &= \|p\| \\ &= \max[|p(0)|, |p(1)|] \\ &= \max[|f(0)|, |f(1)|] \\ &\leq \|f\|, \qquad f \in \mathscr{C}[0, 1]. \end{aligned} \tag{3.9}$$

Hence the value of $\|X\|$ is at most one. Because the function $\{f(x)=1;\, 0 \leq x \leq 1\}$ shows that $\|X\|$ is at least one, it follows that the norm of the approximation operator is equal to one. The norms of several other operators are calculated later, and the work of the next section gives one reason why they are important.

3.2 Lebesgue constants

The norm of an approximation operator is sometimes called the Lebesgue constant of the operator. In particular this term is used when one compares the error of a calculated approximation with the smallest error that can be achieved. The next theorem shows that the value of the norm is of direct relevance to this comparison.

Theorem 3.1

Let $\mathscr{A}$ be a finite-dimensional linear subspace of a normed linear space $\mathscr{B}$, and let X be a linear operator from $\mathscr{B}$ to $\mathscr{A}$ that satisfies the projection condition (3.2). For any f in $\mathscr{B}$, let d^* be the least distance

$$d^* = \min_{a \in \mathscr{A}} \|f - a\| \tag{3.10}$$

from f to an element of $\mathscr{A}$. Then the error of the approximation $X(f)$ satisfies the bound

$$\|f - X(f)\| \leq [1 + \|X\|]d^*. \tag{3.11}$$

Proof. Let p^* be a best approximation from $\mathscr{A}$ to f, which is shown to exist by Theorem 1.2. The projection condition (3.2) and the linearity of X give the equation

$$f - X(f) = (f - p^*) - X(f - p^*). \tag{3.12}$$

It follows from the triangle inequality for norms, and from the definitions of $\|X\|$ and p^*, that the bound

$$\begin{aligned}\|f - X(f)\| &\leq \|f - p^*\| + \|X(f - p^*)\| \\ &\leq [1 + \|X\|]\, \|f - p^*\| \\ &= [1 + \|X\|]d^* \end{aligned} \tag{3.13}$$

is obtained, which is the required result. □

If we apply this theorem to the example given in Section 3.1, where $p = X(f)$ is the linear polynomial that satisfies the conditions (3.6), then we find the bound

$$\|f - X(f)\|_\infty \leq 2 \min_{p \in \mathscr{P}_1} \|f - p\|_\infty. \tag{3.14}$$

Hence the maximum error of the approximation from $\mathscr{P}_1$ to f that is defined by the interpolation conditions (3.6) is never more than twice the least maximum error that can be achieved. Results of this kind often show that the extra work of calculating best approximations is not worthwhile.

If the interpolation method (3.6) is applied to the function

$$f(x) = x^2, \qquad 0 \leq x \leq 1, \tag{3.15}$$

then the calculated approximation is the polynomial $\{p(x) = x; 0 \leq x \leq 1\}$, while the approximation that minimizes the ∞-norm of the error is the function $\{p^*(x) = x - \frac{1}{8}, 0 \leq x \leq 1\}$. This example shows that expression (3.11) can be satisfied as an equality.

One useful application of Theorem 3.1 is to the case when one requires a polynomial approximation p to a function f in $\mathscr{C}[a, b]$ that satisfies the condition

$$\|f - p\|_\infty \leq \varepsilon, \tag{3.16}$$

where ε is a given positive number. The degree of the polynomial is not specified, but it should not be much larger than necessary. Let $\mathscr{A}$ be the space $\mathscr{P}_n$ of polynomials of degree at most n, and let X be a linear operator from $\mathscr{C}[a, b]$ to $\mathscr{A}$ that satisfies condition (3.2). If $X(f)$ is calculated, and if it is found that at a point of the range $[a, b]$ the modulus of the error function $[f - X(f)]$ is larger than $[1 + \|X\|_\infty]\varepsilon$, then it follows from Theorem 3.1 that the degree of p must exceed n. Hence it is possible sometimes to derive useful information about best approximations from simple algorithms. Therefore, when we consider practical algorithms that are linear projections, we usually give some attention to the norm of the approximation operator.

3.3 Polynomial approximations to differentiable functions

Much of the work of this book is given to approximation by polynomials. One could try to justify this specialization by the well-known Weierstrass theorem. It is proved in Chapter 6, and it states that, for any f in $\mathscr{C}[a, b]$ and for anv $\varepsilon > 0$, there exists an algebraic polynomial p that satisfies the condition

$$\|f - p\|_\infty \leq \varepsilon. \tag{3.17}$$