7.3 The characterization theorem and the Haar condition

If the set $\mathscr{A}$ of approximating functions is the space $\mathscr{P}_n$ of algebraic polynomials of degree at most n, then it is rather easy to test whether condition (7.6) can be obtained. We make use of the fact that a function in $\mathscr{P}_n$ has at most n sign changes. Therefore, if the error function $[f(x)-p^*(x)]$ changes sign more than n times as x ranges over $\mathscr{Z}_{\mathrm{M}}$, then p^* is a best approximation. Conversely, if the number of sign changes does not exceed n, then we can choose the zeros of a polynomial in $\mathscr{P}_n$ so that condition (7.6) is satisfied. This result is usually called the minimax characterization theorem, and it is stated formally below.

It is useful to express the theorem in a form that applies to a class of functions that includes polynomials as a special case. The usual way of defining this class is to identify the properties of polynomials that are used in the proof of the characterization theorem. They are the following two conditions:

(1) If an element of $\mathscr{P}_n$ has more than n zeros, then it is identically zero.

(2) Let $\{\zeta_j; j=1, 2, \ldots, k\}$ be any set of distinct points in the open interval (a, b), where $k \leq n$. There exists an element of $\mathcal{P}_n$ that changes sign at these points, and that has no other zeros. Moreover, there is a function in $\mathcal{P}_n$ that has no zeros in $[a, b]$.

The following two properties of polynomials are required later:

(3) If a function in $\mathcal{P}_n$, that is not identically zero, has j zeros, and if k of these zeros are interior points of $[a, b]$ at which the function does not change sign, then the number $(j+k)$ is not greater than n.

(4) Let $\{\xi_j; j=0, 1, \ldots, n\}$ be any set of distinct points in $[a, b]$, and let $\{\phi_i; i=0, 1, \ldots, n\}$ be any basis of $\mathcal{P}_n$. Then the $(n+1)\times(n+1)$ matrix whose elements have the values $\{\phi_i(\xi_j); i=0, 1, \ldots, n; j=0, 1, \ldots, n\}$ is non-singular.

An $(n+1)$-dimensional linear subspace $\mathcal{A}$ of $\mathcal{C}[a, b]$ is said to satisfy the 'Haar condition' if these four statements remain true when $\mathcal{P}_n$ is replaced by the set $\mathcal{A}$. Equivalently, any basis of $\mathcal{A}$ is called a 'Chebyshev set'. Spaces that satisfy the Haar condition are studied in Appendix A. It is proved that properties (**1**), (**3**) and (**4**) are equivalent, and that these properties imply condition (**2**). It is usual to define the Haar condition in terms of the first property. Thus $\mathcal{A}$ satisfies the Haar condition if and only if, for every non-zero p in $\mathcal{A}$, the number of roots of the equation $\{p(x)=0; a \leq x \leq b\}$ is less than the dimension of $\mathcal{A}$.

Theorem 7.2 (Characterization Theorem)

Let $\mathcal{A}$ be an $(n+1)$-dimensional linear subspace of $\mathcal{C}[a, b]$ that satisfies the Haar condition, and let f be any function in $\mathcal{C}[a, b]$. Then p^* is the best minimax approximation from $\mathcal{A}$ to f, if and only if there exist $(n+2)$ points $\{\xi_i^*; i=0, 1, \ldots, n+1\}$, such that the conditions

$$a \leq \xi_0^* < \xi_1^* < \ldots < \xi_{n+1}^* \leq b, \tag{7.17}$$

$$|f(\xi_i^*) - p^*(\xi_i^*)| = \|f-p^*\|_\infty, \qquad i=0, 1, \ldots, n+1, \tag{7.18}$$

and

$$f(\xi_{i+1}^*) - p^*(\xi_{i+1}^*) = -[f(\xi_i^*) - p^*(\xi_i^*)], \qquad i=0, 1, \ldots, n, \tag{7.19}$$

are obtained.

Proof. We let $\mathcal{X}$ be the interval $[a, b]$ in Theorem 7.1. The present theorem is proved in the way that is described in the first paragraph of this section, by making use of the properties (**1**) and (**2**) that are stated above, which hold when $\mathcal{A}$ satisfies the Haar condition. □