2

The uniqueness of best approximations

2.1 Convexity conditions

In order to approximate a point or a function f by an element of a set $\mathscr{A}$, it is usual to choose conditions that define a particular approximation. Best approximation with respect to an appropriate distance function is often suitable, but sometimes there are several best approximations. Some general conditions for uniqueness are given in this chapter, that depend on the convexity of the distance function and the convexity of the set $\mathscr{A}$. Hence it is shown that in many important cases the best approximation is unique, including best approximation with respect to the 2-norm when $\mathscr{A}$ is a linear space. We find, however, that, if the 1-norm or ∞-norm is used, then stronger conditions are required on $\mathscr{A}$ in order to ensure uniqueness.

The set $\mathscr{S}$ of a linear space is convex if, for all s_0 and s_1 in $\mathscr{S}$, the points $\{\theta s_0 + (1-\theta)s_1; 0 < \theta < 1\}$ are also in $\mathscr{S}$. The set is strictly convex if, for all $s_0 \neq s_1$, the points $\{\theta s_0 + (1-\theta)s_1; 0 < \theta < 1\}$ are interior points of $\mathscr{S}$. Thus, it is not possible for the boundary of a strictly convex set to contain a segment of a straight line. The nature of the ideas that are studied in this chapter is suggested by considering the uniqueness of the best approximation if the circles in Figure 1.4 are replaced by balls that are derived from some other norm. Our next theorem shows that these balls are convex sets.

Theorem 2.1

Let $\mathscr{B}$ be a normed linear space. Then, for any $f \in \mathscr{B}$ and for any $r > 0$, the ball

$$\mathscr{N}(f, r) = \{x\colon \|x - f\| \leq r, x \in \mathscr{B}\} \tag{2.1}$$

is convex.

Proof. Let x_0 and x_1 be in $\mathcal{N}(f, r)$. Then the axioms of a norm and the definition (2.1) give the bound

$$\begin{aligned} \|\theta x_0 + (1-\theta)x_1 - f\| &\leq \|\theta x_0 - \theta f\| + \|(1-\theta)x_1 - (1-\theta)f\| \\ &= |\theta|\, \|x_0 - f\| + |1-\theta|\, \|x_1 - f\| \\ &\leq r\{|\theta| + |1-\theta|\} \\ &= r, \; 0 < \theta < 1, \end{aligned} \tag{2.2}$$

which is the required convexity condition. □

It is now easy to prove one of the basic properties of best approximations, which depends on the convexity of the set of approximating functions. This convexity condition holds, of course, when $\mathscr{A}$ is a linear space.

Theorem 2.2

Let $\mathscr{A}$ be a convex set in a normed linear space $\mathscr{B}$, and let f be any point of $\mathscr{B}$ such that there exists a best approximation from $\mathscr{A}$ to f. Then the set of best approximations is convex.

Proof. Let h^* be the error of the best approximation

$$h^* = \min_{a \in \mathscr{A}} \|a - f\|. \tag{2.3}$$

The set of best approximations is the intersection of $\mathscr{A}$ and the ball $\mathcal{N}(f, h^*)$. The theorem follows from the fact that the intersection of two convex sets is convex. □

The uniqueness theorems of the next section require either $\mathscr{A}$ or the norm of the linear space $\mathscr{B}$ to be strictly convex. The norm is defined to be strictly convex if and only if the unit ball centred on the origin, namely $\mathcal{N}(0, 1)$, is strictly convex. Because the general ball (2.1) can be obtained from $\mathcal{N}(0, 1)$ by translation and magnification, strict convexity of the norm implies that the set (2.1) is strictly convex for any f and r.

2.2 Conditions for uniqueness of the best approximation

The two uniqueness theorems that are given below are self-evident if one takes the geometric view of best approximation that is described in Section 1.5. We recall that a ball with centre f is allowed to grow until it touches the set $\mathscr{A}$ of approximating functions, and then the radius of the ball has the value (2.3). The two theorems state that there is only one point of contact between $\mathscr{A}$ and $\mathcal{N}(f, h^*)$, if the boundary of either $\mathscr{A}$ or $\mathcal{N}(f, h^*)$ is curved, and if both sets are convex.

Theorem 2.3

Let $\mathscr{A}$ be a compact and strictly convex set in a normed linear space $\mathscr{B}$. Then, for all $f \in \mathscr{B}$, there is just one best approximation from $\mathscr{A}$ to f.

Proof. Theorem 1.1 shows that there is a best approximation. We continue to let h^* be the error (2.3). Suppose that s_0 and s_1 are different best approximations from $\mathscr{A}$ to f. Because the triangle inequality for norms gives the condition

$$\|\tfrac{1}{2}(s_0+s_1)-f\| \leq \tfrac{1}{2}\|s_0-f\|+\tfrac{1}{2}\|s_1-f\|, \tag{2.4}$$

and because $\mathscr{A}$ is convex, it follows that $\frac{1}{2}(s_0+s_1)$ is also a best approximation, and therefore it satisfies the equation

$$\|\tfrac{1}{2}(s_0+s_1)-f\| = h^*. \tag{2.5}$$

We let λ be the largest number in the interval $0 \leq \lambda \leq 1$ such that the point

$$s = \tfrac{1}{2}(s_0+s_1)+\lambda[f-\tfrac{1}{2}(s_0+s_1)] \tag{2.6}$$

is in $\mathscr{A}$. The value of λ is well-defined because $\mathscr{A}$ is compact. Expressions (2.5) and (2.6) imply the equation

$$\|s-f\| = (1-\lambda)h^*. \tag{2.7}$$

However, h^* is positive because otherwise $s_0 = f = s_1$, and λ is positive because the strict convexity of $\mathscr{A}$ implies that $\frac{1}{2}(s_0+s_1)$ is an interior point of $\mathscr{A}$. It therefore follows from equation (2.7) that $\|s-f\|$ is less than h^*. This contradiction proves the theorem. □

Theorem 2.4

Let $\mathscr{A}$ be a convex set in a normed linear space $\mathscr{B}$, whose norm is strictly convex. Then, for all $f \in \mathscr{B}$, there is at most one best approximation from $\mathscr{A}$ to f.

Proof. Suppose that s_0 and s_1 are different best approximations from $\mathscr{A}$ to f. Because the strict convexity of the norm implies that the set $\mathscr{N}(f, h^*)$ is strictly convex, the point $\frac{1}{2}(s_0+s_1)$ is an interior point of $\mathscr{N}(f, h^*)$, which is the condition

$$\|\tfrac{1}{2}(s_0+s_1)-f\| < h^*. \tag{2.8}$$

This is a contradiction, however, because $\frac{1}{2}(s_0+s_1) \in \mathscr{A}$. The theorem is proved. □

Theorem 2.4 is much more useful to us than Theorem 2.3, because our sets of approximating functions are finite-dimensional linear subspaces.

Therefore it is important to know whether the norm of $\mathscr{B}$ is strictly convex. It is proved in Section 2.4 that the 2-norms in $\mathscr{C}[a, b]$ and in $\mathscr{R}^n$ are strictly convex, but that the 1- and ∞-norms are not. In fact all the p-norms are strictly convex for $1 < p < \infty$.

2.3 The continuity of best approximation operators

When there is a unique best approximation from $\mathscr{A}$ to f for all $f \in \mathscr{B}$, we can regard the best approximation as a function of f. Hence there is a best approximation operator from $\mathscr{B}$ to $\mathscr{A}$, which we call X, and which, incidentally, must be a projection. It is shown in this section that often the operator X is continuous. This result is important to computer calculations, because, if it does not hold, then the effect of computer rounding errors in the definition of f may cause substantial changes to the calculated approximation.

Theorem 2.5

Let $\mathscr{A}$ be a compact set in a metric space $\mathscr{B}$, such that for every f in $\mathscr{B}$ there is only one best approximation in $\mathscr{A}$, $X(f)$ say. Then the operator X, defined by the best approximation condition, is continuous.

Proof. If the theorem is false, there exists a sequence of points $\{f_i; i = 1, 2, 3, \ldots\}$ in $\mathscr{B}$ that converges to a limit, f say, such that the sequence $\{X(f_i); i = 1, 2, 3, \ldots\}$ in $\mathscr{A}$ fails to converge to $X(f)$. Therefore, by compactness, the second sequence has a limit point, a^* say, that is in $\mathscr{A}$ but that is not equal to $X(f)$. It suffices to show that both a^* and $X(f)$ are best approximations to f, for then we have a contradiction that proves the theorem.

Therefore we consider the distance $d(a^*, f)$, and, by applying the triangle inequality (1.1) twice, we deduce the bound

$$d(a^*, f) \leq d(a^*, X(f_i)) + d(X(f_i), f_i) + d(f_i, f). \tag{2.9}$$

Moreover, the definition of $X(f_i)$ gives the relation

$$\begin{aligned} d(X(f_i), f_i) &\leq d(X(f), f_i) \\ &\leq d(X(f), f) + d(f, f_i), \end{aligned} \tag{2.10}$$

where the last line makes use of the triangle inequality again. Now, for any $\varepsilon > 0$, there exists i such that the conditions

$$d(a^*, X(f_i)) \leq \tfrac{1}{3}\varepsilon \tag{2.11}$$

and

$$d(f_i, f) \leq \tfrac{1}{3}\varepsilon \tag{2.12}$$

hold. It follows from expressions (2.9) and (2.10) that the bound

$$d(a^*, f) \leq d(X(f), f) + \varepsilon \tag{2.13}$$

is obtained. Since ε can be arbitrarily small, a^* is a best approximation from $\mathscr{A}$ to f, which is the required contradiction. □

By applying the technique that is used in the proof of Theorem 1.2, it can be shown that the following theorem is true also. The proof is left as an exercise.

Theorem 2.6

If $\mathscr{A}$ is a finite-dimensional linear space in a normed linear space $\mathscr{B}$, such that for every f in $\mathscr{B}$ there is only one best approximation in $\mathscr{A}$, $X(f)$ say, then the operator X, defined by the best approximation condition, is continuous. □

The last theorem is directly relevant to the approximation problems that are studied in later chapters. Note that it provides additional motivation for giving attention to the uniqueness of best approximations.

2.4 The 1-, 2- and ∞-norms

The method that we use to prove that the 2-norm is strictly convex in $\mathscr{C}[a, b]$ and $\mathscr{R}^n$ makes use of scalar products. It is well known that the scalar product of y and z in $\mathscr{R}^n$ has the value

$$(y, z) = \sum_{i=1}^{n} y_i z_i, \tag{2.14}$$

and in $\mathscr{C}[a, b]$ the scalar product of the functions f and g is the expression

$$(f, g) = \int_a^b f(x) g(x) \, dx. \tag{2.15}$$

It is important to note that (f, f) is equal to $\|f\|_2^2$. Further, the identity

$$\|f + g\|_2^2 = \|f\|_2^2 + 2(f, g) + \|g\|_2^2 \tag{2.16}$$

is obtained, either when f and g are in $\mathscr{C}[a, b]$, or when they are in $\mathscr{R}^n$. In fact it holds for all Hilbert spaces, but, if the reader has not met Hilbert spaces before, it is sufficient for him to recognise that equation (2.16) is valid both for $\mathscr{C}[a, b]$ and for $\mathscr{R}^n$. We note also that the scalar product (f, g) is linear in f and in g.

Theorem 2.7

The 2-norm of the linear space $\mathscr{B}$ is strictly convex when $\mathscr{B}$ is either $\mathscr{C}[a, b]$ or $\mathscr{R}^n$.

Proof. We let f and g be any two distinct points of $\mathscr{B}$ such that $\|f\|_2=\|g\|_2=1$. It is sufficient to prove that the bound

$$\|\theta f+(1-\theta)g\|_2<1 \tag{2.17}$$

is satisfied for all $0<\theta<1$. The identity

$$\begin{aligned}\|\theta f+(1-\theta)g\|_2^2&+\theta(1-\theta)\|f-g\|_2^2\\ &=\theta^2+2\theta(1-\theta)(f,g)+(1-\theta)^2+\theta(1-\theta)[1-2(f,g)+1]\\ &=1,\end{aligned} \tag{2.18}$$

which holds for all values of θ, gives the required inequality (2.17). □

It has been stated already that the 1- and ∞-norms in $\mathscr{C}[a,b]$ and in $\mathscr{R}^n$ are not strictly convex, and now this statement is proved. We also wish to find out whether best approximations from linear subspaces are always unique. If we prove first that the norms are not strictly convex, then Theorem 2.4 does not answer the uniqueness question. If instead, however, we can demonstrate that a best approximation from a linear subspace of a normed linear space is not unique, then we may deduce from Theorem 2.4 that the norm is not strictly convex. We give examples of this kind. In each one there is a linear subspace $\mathscr{A}$ and a point f such that the best approximation from $\mathscr{A}$ to f is not unique, where $\mathscr{A}$ and f are contained in either $\mathscr{C}[a,b]$ or in $\mathscr{R}^n$, and where the accuracy of the approximation is measured either by the 1-norm or by the ∞-norm.

When the 1-norm is used in $\mathscr{C}[-1,1]$, we let f be the constant function whose value is one, and we let $\mathscr{A}$ be the one-dimensional linear space that contains all functions of the form

$$a(x)=\lambda x, \qquad -1\leqslant x\leqslant 1, \tag{2.19}$$

where λ is a parameter. It is straightforward to derive the equation

$$\min_{a\in\mathscr{A}}\int_{-1}^{1}|f(x)-a(x)|\,dx=2, \tag{2.20}$$

and to show that the minimum value is obtained when λ is in the range $-1\leqslant\lambda\leqslant 1$. Hence the best approximation is not unique.

This example for the 1-norm is extended to $\mathscr{R}^n$ $(n\geqslant 2)$ by dividing the interval $[-1,1]$ by the points $-1=x_1<x_2<\ldots<x_n=1$, which are equally spaced

$$x_{i+1}-x_i=2/(n-1), \qquad i=1,2,\ldots,n-1. \tag{2.21}$$

We evaluate the function f that we had before at these points to give a vector $f\in\mathscr{R}^n$. Moreover, corresponding to equation (2.19), we let $a\in\mathscr{A}\subset\mathscr{R}^n$ be the vector whose components have the values

$$a_i=\lambda x_i, \qquad i=1,2,\ldots,n, \tag{2.22}$$

where λ is still a parameter. Now, instead of equation (2.20), we find the expression

$$\min_{a \in \mathscr{A}} \sum_{i=1}^{n} |f_i - a_i| = n, \tag{2.23}$$

and again the minimum value is obtained for all values of λ in the range $-1 \leqslant \lambda \leqslant 1$.

For the ∞-norm in $\mathscr{C}[-1, 1]$, we again let f be the constant function whose value is one, but now we let $\mathscr{A}$ be the one-dimensional linear space that contains functions of the form

$$a(x) = \lambda(1 + x), \qquad -1 \leqslant x \leqslant 1. \tag{2.24}$$

We deduce the equation

$$\min_{a \in \mathscr{A}} \|f - a\|_\infty = 1, \tag{2.25}$$

and we find that the function (2.24) is a best approximation if and only if λ satisfies the condition

$$0 \leqslant \lambda \leqslant 1. \tag{2.26}$$

Hence we have non-uniqueness once more. We extend the example to $\mathscr{R}^n$ in the way described in the previous paragraph. The components of $f \in \mathscr{R}^n$ are the same as before, but, because of equation (2.24), the components of $a \in \mathscr{A}$ have the values

$$a_i = \lambda(1 + x_i), \qquad i = 1, 2, \ldots, n, \tag{2.27}$$

instead of the values (2.22). The range of values of λ that give a best approximation from $\mathscr{A}$ to f is still the range (2.26).

The reader is advised to draw figures that show the non-uniqueness of the best approximation in these four examples. It should be noted also that the examples illustrate the usefulness of Theorem 2.2.

In many important cases, in particular when the normed linear space is $\mathscr{C}[a, b]$, when the norm is either the 1-norm or the ∞-norm, and when $\mathscr{A}$ is the space $\mathscr{P}_n$ of algebraic polynomials of degree at most n, then the best approximation is unique for all f in $\mathscr{C}[a, b]$. This statement is proved later. The purpose of the examples, therefore, is to show that, if $\mathscr{A}$ is a linear subspace of a normed linear space, whose norm is not strictly convex, then the uniqueness of best approximations depends on properties of $\mathscr{A}$ and f.